For $\emptyset \neq A \subseteq \mathbb{R}$, let A^c denote the set of all cluster (= accumulation) points w.r.t. A, i.e. $x \in A^c$ iff $\forall \delta > 0\ \exists\, a \in A \setminus \{x\}$ s.t. $0 < |x-a| < \delta$.

Th (Characterization). For $x \in \mathbb{R}$, TFSAE:

(i) $x \in A^c$

(ii) the distance $\mathrm{dist}(x, A\setminus\{x\}) := \inf\{|x-a| : a \in A\setminus\{x\}\} = 0$

(iii) $\forall n \in N\ \exists\, a_n \in A \setminus \{x\}$ s.t. $0 < |x - a_n| < 1/n$

(iv) $\exists$ a seq (a_n) in $A \setminus \{x\}$ s.t. $\lim_n a_n = x$

Let $\emptyset \neq A \subseteq \mathbb{R}$, $x_0 \in A^c$ and $f: A \to \mathbb{R}$, $l \in \mathbb{R}$. We say that $f(x)$ converges to l as x converges to x_0 if $\forall \varepsilon > 0\ \exists\, \delta > 0$ s.t.

(*) $|f(x) - l| < \varepsilon$ whenever $x \in A \setminus \{x_0\}$ & $|x - x_0| < \delta$.

Th1 (Uniqueness)

Th2 (Sequential Criterion – so $\lim_{x \to x_0} f(x)$ notation OK); let $l \in \mathbb{R}$. TFSAE:

(i) $f(x) \to l$ as $x \to x_0$

(ii) $f(x_n) \to l$ whenever (x_n) is a seq in $A \setminus \{x_0\}$ convergent to x_0

Th2* (NOT yet fix $l \in \mathbb{R}$). TFSAE:

(i*) $\lim_{x \to x_0} f(x)$ exists in $\mathbb{R}$

(ii*) $\lim_n f(x_n)$ exists in $\mathbb{R}$ whenever (x_n) is a seq in $A \setminus \{x_0\}$ conv. to x_0.

Note. (i*) $\Rightarrow$ (ii*) certainly follows from (i) $\Rightarrow$ (ii) of Th 2. But, for (ii*) $\Rightarrow$ (i*), we must show that all $(f(x_n))$ have the <u>same</u> limit when $(x_n) \to x_0$ with each $x_n \in A \setminus \{x_0\}$. To prove this, let (x_n'), (x_n'') be seq in $A \setminus \{x_0\}$ convergent to x_0. By (ii*),

Let $\lim_n f(x_n') = l'$ & $\lim_n f(x_n'') = l''$. Let
(x_n) be the "alternate seq":

$$x_1', x_1'', x_2', x_2'', x_3', x_3'', \ldots$$

i.e.

$$x_{2n-1} = x_n' \quad \& \quad x_{2n} = x_n'' \quad \forall n$$

Then (x_n) is a seq in $A \setminus \{x_0\}$ convergent to x_0 and it follows from (ii) that $\lim_n f(x_n)$ exists in $\mathbb{R}$ so $\lim_n f(x_n') = \lim_n f(x_n'')$ ($\because$ being subsequences), as required to show.

Th3 (Divergence Th). (i) $\Leftrightarrow$ (ii) $\Leftarrow$ (iii), where

(i) $\lim_{x \to x_0} f(x)$ not exist (in $\mathbb{R}$)

(ii) $\exists$ a seq (x_n) in $A \setminus \{x_0\}$ with $x_n \to x_0$ but $\lim_n f(x_n)$ not exist (in $\mathbb{R}$).

(iii) $\exists$ sequences $(x_n), (x_n')$ in $A \setminus \{x_0\}$ ~~with~~ convergent to x_0 such that $\lim_n f(x_n) = l \neq l' = \lim_n f(x_n')$.

Also, if f is a bounded function, then (i) $\Rightarrow$ (iii).

proof. The last part is by B-W argument.

Th 4. (Limit is a local property). Suppose $\exists\, \delta > 0$ s.t. $f = g$ on $(A \setminus \{x_0\}) \cap V_\delta(x_0)$. Then

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x)$$

(if one of the limits exist). Moreover, if $\lim_{x \to x_0} f(x)$ exists, then $\exists\, \delta_0 > 0$ s.t. $\{f(x) : x \in (A \setminus \{x_0\}) \cap (V_{\delta_0}(x_0))\}$ is bounded.

Th 5 (Order-Preserving). Let

$$\alpha < l := \lim_{x \to x_0} f(x) < \beta$$

Then $\exists\, \delta > 0$ s.t.

$$(\#) \quad \alpha < f(x) < \beta \quad \forall\, x \in (A \setminus \{x_0\}) \cap (V_\delta(x_0))$$

Pf. Let $\varepsilon = \min\{\beta - l, l - \alpha\}$.

Then $\varepsilon > 0$ and $\cancel{V_\varepsilon(l)} = (l-\varepsilon, l+\varepsilon) \subseteq (\alpha, \beta)$. Since $l = \lim_{x \to x_0} f(x)$. For this ε, $\exists\, \delta > 0$ s.t.

$\#$ $l - \varepsilon < f(x) < l + \varepsilon$ whenever $x \in (A \setminus \{x_0\}) \cap (V_\delta(x_0))$

which implies $(\#)$.

Remark. With suitable interpretation, α may be replaced by $-\infty$ and/or β by $+\infty$.

Cor. Suppose, for some $\delta > 0$,

$$f(x) \geq \beta \quad \forall\, x \in (A \setminus \{x_0\}) \cap (V_\delta(x_0))$$

Then, if $\lim_{x \to x_0} f(x) = l$ exists, one has $l \geq \beta$.

(Here, again, you need the assumption that $x_0 \in A^c$).

Th 6. Let $\lim_{x \to x_0} f(x) = l$. Then f is "locally bounded around x_0", and (see Th 4)

(i) $\lim_{x \to x_0} |f(x)| = |l|$

(ii) Assuming further that $l \neq 0$, $\exists\, \delta > 0$ s.t.

$$\frac{1}{2}|l| < |f(x)| < \frac{3}{2}|l| \quad \forall\, x \in (A \setminus \{x_0\}) \cap (V_\delta(x_0)).$$

Pf The proof for (i) is easy. Then (ii) follows from (i) and Th 5 as

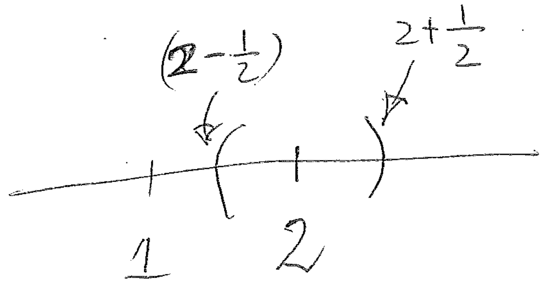
$$\frac{|l|}{2} < \lim_{x \to x_0} |f(x)| = |l| < \frac{3|l|}{2}$$

Th 7. (Squeeze Th) Let $f(x) \leq g(x) \leq h(x)$ $\forall x \in A \setminus \{c\}$, and suppose $\lim_{x \to x_0} f(x) = l = \lim_{x \to x_0} h(x)$. Then $\lim_{x \to x_0} g(x) = l$.

$$\lim_{x\to 2}\frac{x^3-4}{x^2-1}=\frac{4}{3}$$

$$\frac{x^3-4}{x^2-1}-\frac{4}{3}=\frac{3x^3-12-4x^2+4}{3(x^2-1)}=\frac{3x^3-4x^2-8}{3(x^2-1)}$$

$$=\frac{(x-2)(3x^2+2x+4)}{3(x^2-1)}$$



Let $\varepsilon>0$. Take $\delta>0$ s.t

$\delta\le 1/2$ and

$\delta\le\frac{15\varepsilon}{68}$ (that is $\delta\le\min\{\frac{1}{2},\frac{15\varepsilon}{148}\}$)

Let $x\in V_\delta(2)$ ($\frac{3}{2}\le 2-\delta<x<2+\delta<2+1/2<3$). Then

① $|x-2|\le|x|+2<5,\ \delta$

② $|3x^2+2x+4|=3x^2+2x+4\le 27+2\times 3+4=37$

③ $|x^2-1|\ge x^2-1\ge(\frac{3}{2})^2-1=\frac{5}{4}\ (>1)$

and hence

$$\left|\frac{x^3-4}{x^2-1}-\frac{4}{3}\right|\le\frac{|x-2|\,|3x^2+2x+4|}{3\,|x^2-1|}\le\frac{4}{15}\times\cancel{5}\times 37\times\delta\le\varepsilon$$

if $\delta<\frac{15\varepsilon}{4\times 37}$